THE RELATIVE GROWTH RATE FOR PARTIAL QUOTIENTS

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Abstract. The rate of growth of the partial quotients of an irrational number is studied relative to the rate of approximation of the number by its convergents. The focus is on the Hausdorff dimension of exceptional sets on which different growth rates are achieved.

In this note we look at the rate of growth of the partial quotients a_i of the irrational number

$$x = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

relative to the rate at which x is approximated by its rational convergents.

For $x \in (0,1)$ irrational, let $\{\frac{p_n}{q_n}\}$ be the sequence of rational convergents given by the continued fraction expansion of x [6]. It follows from classical results of Khinchin and Lévy [3] that for almost all x

$$\lim_{n\to\infty} \frac{\log a_n}{n} = 0 \quad \text{and} \quad \lim_{n\to\infty} -\frac{\log |x - \frac{p_n}{q_n}|}{n} = \frac{\pi^2}{6\log 2}. \tag{1}$$

Consequently, for almost all $x \in (0,1)$

$$\lim_{n\to\infty} \frac{\log a_{n+1}}{\log |x - \frac{p_n}{q_n}|} = 0. \tag{2}$$

Here we study the Hausdorff dimension of exceptional sets on which the limit (2) either does not exist or is different from zero. Similar, non-overlapping, problems are considered in [8] using more sophisticated methods of multifractal analysis.

We shall write $\mathrm{Dim}_H X$ for the Hausdorff dimension of a set $X \subset [0,1]$ and $\mathcal{H}^s(X)$ for the Hausdorff s-dimensional measure of X [4].

Let

$$\mathscr{F}(z) = \left\{ x \in (0,1) \,:\, \limsup_{n\to\infty} \frac{-\log a_{n+1}}{\log |x - \frac{p_n}{q_n}|} = z \right\}. \tag{3}$$

Theorem 1. *For $0 < z \leq 1$, $Dim_H \mathscr{F}(z) = 1 - z$ and $\mathcal{H}^{1-z}(\mathscr{F}(z)) = \infty$. If $z \notin [0,1]$ then $\mathscr{F}(z) = \emptyset$.*

By an earlier remark, $\mathscr{F}(0)$ is a set of Lebesgue measure 1.

There is an alternative characterization of the problem in terms that compare the rate of growth of the denominators of the convergents to the rate at which they

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approximate x. It is, in its own right an interesting way to look at the problem. For $\alpha \in \mathbb{R}$ define the set

$$\mathscr{G}(\alpha) = \left\{ x \in (0,1) \,:\, \limsup_{n\to\infty} \frac{\log q_n^2}{\log |x - \frac{p_n}{q_n}|} = \alpha \right\}.$$

Lemma 1. *When $z \in [0,1]$, $\mathscr{F}(z) = \mathscr{G}(z-1)$.*

Proof. Define the approximation constants $\theta_n(x) = q_n|q_nx - p_n|$. From the classical theory of continued fractions we have

$$\theta_n(x) = \frac{1}{a'_{n+1} + \frac{q_{n-1}}{q_n}} \tag{4}$$

where $a'_{n+1} = a_{n+1} + [a_{n+2}, \ldots]$ [6]. Therefore

$$\limsup_{n\to\infty} \frac{-\log a_{n+1}}{\log |x - \frac{p_n}{q_n}|} = \limsup_{n\to\infty} \frac{\log \theta_n(x)}{\log |x - \frac{p_n}{q_n}|} = \limsup_{n\to\infty} \frac{\log q_n^2}{\log |x - \frac{p_n}{q_n}|} + 1.$$

□

At this point it is an easy matter to show that $\mathscr{F}(1)$ is an infinite set and therefore $\mathcal{H}^0(\mathscr{F}(1)) = \infty$. In fact, if one chooses the partial quotients so that $q_n^{2n} < a_{n+1}$, then using (4) it follows that the limit in (2) is equal to 1. In order to complete the proof of Theorem 1, we shall work with the alternative formulation suggested by the lemma and prove

Theorem 2. *For $\alpha \in (-1,0]$, $Dim_H \mathscr{G}(\alpha) = |\alpha|$ and $\mathcal{H}^{|\alpha|}(\mathscr{G}(\alpha)) = \infty$. If $\alpha \notin [-1,0]$, then $\mathscr{G}(\alpha) = \emptyset$.*

The set $\mathscr{G}(-1) = \mathscr{F}(0)$ has lebesgue measure 1. Interestingly, the results in [8] imply that if the second limit in (1) exists for a number x (not necessarily taking the value given in (1)), then $x \in \mathscr{G}(-1)$.

The last sentence of Theorem 2 is elementary and is a consequence of the following basic property of the convergents [6]

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n^2}.$$

The main tool in the proof of Theorem 2 is Jarnik's "zero-infinity" law [1, 2, 7]. We need to establish some notation and reframe the problem so that Jarnik's Theorem will apply.

The abbreviation FIM will be used in place of the phrase, "for infinitely many." Given $\tau \in (-1,0)$, and $0 \le \epsilon < |\tau|$, define

$$\psi_{(\tau,\epsilon)}(r) = r^{\frac{2}{\tau+\epsilon}}.$$

Consider the related equation

$$|x - \frac{p}{q}| < \psi_{(\tau,\epsilon)}(q) = q^{\frac{2}{\tau+\epsilon}} \tag{5}$$

and the set

$$W(\psi_{(\tau,\epsilon)}) = \left\{ x \in [0,1] : |x - \frac{p}{q}| < q^{\frac{2}{\tau+\epsilon}} \text{ FIM } \frac{p}{q} \in \mathbb{Q} \right\}.$$

We are not interested in just any rationals but rather in the convergents. Define

$$W^*(\psi_{(\tau,\epsilon)}) = \left\{ x \in [0,1] \; : \; |x - \frac{p_n}{q_n}| < q_n^{\frac{2}{\tau+\epsilon}} \text{ FIM convergents } \frac{p_n}{q_n} \text{ of } x \right\}.$$

Observe that when q is sufficiently large,

$$q^{\frac{2}{\tau+\epsilon}} < \frac{1}{2} q^{-2}. \tag{6}$$

If $\frac{p}{q}$ satisfies inequalities (5) and (6) then it is a convergent of x [6]. Therefore, except for finitely many rationals, $\frac{p}{q}$ satisfies (5) if and only if it is a convergent of x. Consequently, $W(\psi_{(\tau,\epsilon)}) = W^*(\psi_{(\tau,\epsilon)})$.

Combining the last observation with a simple manipulation of equation (5) yields

$$W(\psi_{(\tau,\epsilon)}) = \left\{ x \in [0,1] \; : \; \frac{\log q_n^2}{\log |x - \frac{p_n}{q_n}|} > \tau + \epsilon \text{ FIM convergents } \frac{p_n}{q_n} \text{ of } x \right\}.$$

It is therefore clear that for $-1 < \tau < \alpha \leq 0$

$$W(\psi_{(\tau,0)}) \supset \mathscr{G}(\alpha). \tag{7}$$

Now we turn to the computation of Hausdorff dimension.

Lemma 2. *For any $\tau \in (-1, 0)$, $dim_H W(\psi_{(\tau,0)}) = |\tau|$, $\mathcal{H}^{|\tau|}(W(\psi_{(\tau,0)})) = \infty$ and for $\epsilon > 0$, $\mathcal{H}^{|\tau|}(W(\psi_{(\tau,\epsilon)})) = 0$*

Proof. If $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function, then a basic version of Jarnik's Theorem [2] says that for $s \in [0, 1)$

$$\mathcal{H}^s(W(\psi_{(\tau,\epsilon)})) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} r(\psi(r))^s < \infty \\ \infty & \text{if } \sum_{r=1}^{\infty} r(\psi(r))^s = \infty. \end{cases}$$

When $\psi(r) = \psi_{(\tau,\epsilon)}$, the series' involved are easy to analyze and it follows that for $\tau \in (-1, 0)$ and $0 \leq \epsilon < |\tau|$,

$$\mathcal{H}^s(W(\psi_{(\tau,\epsilon)})) = \begin{cases} 0 & \text{for } s > -\tau - \epsilon. \\ \infty & \text{for } s \leq -\tau - \epsilon \end{cases}$$

From this we conclude that $\dim_H W(\psi_{(\tau,0)}) = |\tau|$ and moreover, that $W(\psi_{(\tau,0)})$ has infinite $|\tau|$-measure. Also, when $\epsilon > 0$ the sets $W(\psi_{(\tau,\epsilon)})$ have $|\tau|$-measure zero. □

Proof of Theorem 2. First, it follows from the inclusion (7) and Lemma 2 that

$$\dim_H \mathscr{G}(\alpha) \leq \dim_H W(\psi_{(\tau,0)}) = |\tau| \tag{8}$$

for all $-1 < \tau < \alpha \leq 0$. In particular, this gives $\dim_H \mathscr{G}(0) = 0$

Now suppose $\alpha \in (-1, 0)$ and pick $k < 0$ so that $\frac{1}{k} < |\alpha|$. Define the set

$$E(\alpha) = W(\psi_{(\alpha,0)}) \setminus \left[\bigcup_{n=k}^{\infty} W(\psi_{(\alpha,\frac{1}{n})}) \right].$$

It is clear that $E(\alpha) \subset \mathscr{G}(\alpha)$. Furthermore, applying Lemma 2, we see that $\dim_H E(\alpha) = |\alpha|$ and $\mathcal{H}^{|\alpha|}(E(\alpha)) = \infty$. Thus,

$$|\alpha| = \dim_H E(\alpha) \leq \dim_H \mathscr{G}(\alpha). \tag{9}$$

Together equations (8) and (9) allow us to conclude that $\dim_H \mathscr{G}(\alpha) = |\alpha|$. Since $E(\alpha)$ has infinite $|\alpha|$-measure, so must the larger set $\mathscr{G}(\alpha)$ □

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